Some Theorems for General Equilibrium Analysis

Theorem 1: Let S be a subset of $\mathbb{R}^l$. If S is nonempty, compact, and convex, and if $f\colon S \to S$ is continuous, then f has a fixed point.

Theorem 2: Let S be a subset of $\mathbb{R}^l$ and let $f\colon S \to\!\!\to S$ be a correspondence. If S is nonempty, compact, and convex, and if f has a closed graph and is nonempty-valued and convex-valued, then f has a fixed point.

Theorem 3: Let $\mathcal{P} \subseteq \mathbb{R}^l$ be a set of price-lists and let $z\colon \mathcal{P} \to \mathbb{R}^l$ be an excess demand function defined on $\mathcal{P}$. If z is continuous and satisfies Walras' Law, and if $\mathcal{P}$ includes the unit simplex in $\mathbb{R}^l$, then an equilibrium of z exists.

Theorem 4: Let $E = ((u^i, \mathring{x}^i))_1^n$ be an economy. If each consumer $(u^i, \mathring{x}^i)$ satisfies the conditions
(a) u^i is continuous, increasing, and quasi-concave, and
(b) $\mathring{x}^i_k > 0, \quad k = 1, ..., l,$
then E has a Walrasian equilibrium.

Theorem 5: If every $u^i : \mathbb{R}^l_+ \to \mathbb{R}$ is continuous and strictly increasing, then an allocation $\hat{\mathbf{x}}$ is Pareto efficient for an economy with resources $\mathring{\mathbf{x}} \in \mathbb{R}^l_+$ and consumers $(u^i)_1^n$ if and only if it is a solution of the problem

$$\begin{aligned} &\max_{(x^i_k)\in\mathbb{R}^{nl}_+} u^1(\mathbf{x}^1) \\ \text{subject to} \quad & x^i_k \geqq 0, \quad i = 1, ..., n, \quad k = 1, ..., l \\ & \sum_{i=1}^n x^i_k \leqq \mathring{x}_k, \quad k = 1, ..., l \\ & u^i(\mathbf{x}^i) \geqq u^i(\hat{\mathbf{x}}^i), \quad i = 2, ..., n. \end{aligned} \tag{P-Max}$$

Theorem 6: If every u^i is continuous and locally nonsatiated, then an interior allocation $\hat{\mathbf{x}}$ is Pareto efficient for the economy $(u^i, \mathring{\mathbf{x}}^i)_1^n$ if and only if it is a solution of the problem (P-Max).

Theorem 7: If every u^i is strictly increasing, quasiconcave, and differentiable, then an interior allocation $\hat{\mathbf{x}}$ is Pareto efficient for the economy $(u^i, \mathring{\mathbf{x}}^i)_1^n$ if and only if it satisfies $\sum_1^n \hat{\mathbf{x}}^i = \sum_1^n \mathring{\mathbf{x}}^i$ and

$$MRS^1_{kk'} = ... = MRS^i_{kk'} = ... = MRS^n_{kk'}. \tag{EqualMRS}$$

for any two goods k and k'.

Theorem 8: If every u^i is strictly increasing, quasiconcave, and differentiable, then an allocation $\hat{\mathbf{x}}$ is Pareto efficient for the economy $(u^i, \mathring{\mathbf{x}}^i)_1^n$ if and only if it satisfies $\sum_1^n \hat{\mathbf{x}}^i = \sum_1^n \mathring{\mathbf{x}}^i$ and for any two consumers i, i' and any two goods k, k':

(A) if $x_k^i > 0$ and $x_{k'}^{i'} > 0$, then $MRS_{kk'}^i \geqq MRS_{kk'}^{i'}$.

(B) if $x_{k'}^i > 0$ and $x_k^{i'} > 0$, then $MRS_{kk'}^i \leqq MRS_{kk'}^{i'}$.

Theorem 9: If an allocation $\hat{\mathbf{x}} \in \mathbb{R}_+^{nl}$ is a solution of the problem

$$\begin{aligned} &\max_{(x_k^i) \in \mathbb{R}_+^{nl}} W(\mathbf{x}) = \textstyle\sum_{i=1}^n \alpha_i u^i(\mathbf{x}^i) \\ &\qquad \text{subject to} \quad x_k^i \geqq 0, \quad i = 1, ..., n, \quad k = 1, ..., l \\ &\qquad\qquad \sum_{i=1}^n x_k^i \leqq \mathring{x}_k, \quad k = 1, ..., l \end{aligned} \tag{W-Max}$$

for some numbers $\alpha_1, ..., \alpha_n > 0$, then $\hat{\mathbf{x}}$ is a Pareto allocation for the economy $(u^i, \mathring{\mathbf{x}}^i)_1^n$.

Theorem 10: If the alternative $\hat{x}$ is a solution of the problem

$$\max_{x \in X} W(x) = \textstyle\sum_{i=1}^n \alpha_i u^i(x), \tag{1}$$

for some numbers $\alpha_1, ..., \alpha_n > 0$, then $\hat{x}$ is Pareto efficient in X.

Theorem 11: Let $E = ((u^i, \mathring{\mathbf{x}}^i))_1^n$ be an economy for which each utility function is continuously differentiable, strictly quasiconcave, and strictly increasing. If $(\hat{\mathbf{p}}, (\hat{\mathbf{x}}^i)_1^n)$ is a Walrasian equilibrium for E, then $(\hat{\mathbf{x}}^i)_1^n$ is Pareto efficient for E.

Theorem 12: Let $(\hat{\mathbf{x}}^i)_1^n$ be a Pareto efficient allocation for an economy with total endowment $\mathring{\mathbf{x}} \in \mathbb{R}_{++}^l$ and with utility functions $u^1, ..., u^n$, each of which is continuously differentiable, strictly quasiconcave, and strictly increasing. For each i, let $\mathring{\mathbf{x}}^i = \hat{\mathbf{x}}^i$. Then there is a price-list $\hat{\mathbf{p}}$ for which $(\hat{\mathbf{p}}, (\hat{\mathbf{x}}^i)_1^n)$ is a Walrasian equilibrium of the economy $((u^i, \mathring{\mathbf{x}}^i))_1^n$.